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A PRACTICAL LOOK AT TIME INTEGRATORS FOR DYNAMICAL SYSTEMS*

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The solution of dynamic problems modeled by either finite elements, finite differences or boundary discretization approaches ultimately reduces to a system of discrete dynamic equations in the form

$$\underset{\sim}{M} \cdot \ddot{\underset{\sim}{Y}} + \underset{\sim}{C} \cdot \dot{\underset{\sim}{Y}} + \underset{\sim}{K} \cdot \underset{\sim}{Y} = \underset{\sim}{R}(t)$$

These equations can be solved by either direct integration approaches, modal analysis, Laplace transforms, method of characteristics, etc. One of the most attractive approaches involves the direct numerical integration of these equations. The major problem is the choice of a proper time step to achieve both a stable and accurate solution.

The purpose of this paper is to examine the performance of several classical integrators used in the solution of dynamical systems with various stiffness characteristics. Specifically, the second order central difference technique [1], Newmark's technique [1], Runge-Kutta methods [2], Gear's method [2] are compared. The performance of these algorithms on systems with various stiffness ratios is of particular importance.

The results demonstrate that Newmark's technique is the worst of the methods studied. Even though the method is unconditionally stable, the time steps required for an accurate solution far exceed the explicit techniques. The second order central difference approach is preferable both from the standpoints of computational efficiency and accuracy.

Historically, high order Runge-Kutta methods have been abandoned due to the extreme amounts of computation time required for each time step [1]. This conclusion, while true for problems in which the stiffness ratio is near unity, breaks down for stiff problems. The results demonstrate that the Runge-Kutta-Gill method (a fourth order method) is computationally superior to the second order methods for any degree of stiffness. Gear's method (of various orders) is also compared. These methods perform well for some problems and not for others.

The results demonstrate that the direct integration of dynamical systems is best approached using Runge-Kutta methods with orders chosen based on the stiffness of the problem. Implementation of these techniques for Finite Element spatial discretizations is discussed.

REFERENCES

[1] K. J. Bathe, Finite Element Procedures in Engineering Analysis, Prentice Hall, Inc., Englewood Cliffs, NJ, 1982.

[2] J. D. Lambert, Computational Methods in Ordinary Differential Equations, John Wiley & Sons, London, 1973.

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Dynamic solid mechanics problems are routinely solved by spatially discretizing the continuum using either finite elements or finite difference approximations. The result is a dynamical system of discrete coupled vibration equations (which may, in fact be nonlinear, if the governing continuum equations are nonlinear). To solve the system, one must either employ modal techniques or direct time integration techniques. This study addresses the direct time integration approach.

Two prototype dynamical systems were studied in this work: a stiff, undamped system, and a stiff, damped system (with sub-critical damping). The specific undamped problem studied was

$$\ddot{Y}_1 = -\omega_0^2 Y_1 + \omega_0^2 (1 - R^2) Y_2$$

$$\ddot{Y}_2 = -R^2 \omega_0^2 Y^2 \qquad (1)$$

$$\omega_0 = 2\pi$$

with initial conditions

$$Y_1 (t = 0) = 0 = Y_2 (t = 0)$$

$$\dot{Y}_1 (t = 0) = V_1 ; \quad \dot{Y}_2 (t = 0) = V_2 \qquad (2)$$

The specific underdamped problem studied was

$$\begin{bmatrix} \ddot{Y}_1 \\ \ddot{Y}_2 \end{bmatrix} + \begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} \dot{Y}_1 \\ \dot{Y}_2 \end{bmatrix} + \begin{bmatrix} \omega_0^2 & \omega_0^2(R^2-1) \\ 0 & R^2\omega_0^2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = 0 \quad (3)$$

$$\omega_0 = 2\pi$$

with initial conditions

$$\begin{aligned} &Y_1\ (t = 0) = 0 = Y_2\ (t = 0) \\ &\dot{Y}_1\ (t = 0) = V_1\ ; \quad \dot{Y}_2\ (t = 0) = V_2 \end{aligned} \quad (4)$$

These equations were solved using the six integrators discussed below.

Many time integrators have been employed for solving systems like the above examples. Relatively little has been done, however, to truly investigate the convergence and performance of these integrators for practical examples. Much theoretical work has been done for trivial uncoupled systems, however, the need is for practical guidelines for real situations. It is important to recognize that the problem of stability as well as accuracy plays a role in convergence. It is not, however, the sole important parameter for stiff systems (as is usually assumed). As will be demonstrated, unconditionally stable integrators often require smaller step size and longer computation times than conditionally stable integrators of the same order.

The first integrator used is the second order central difference operator. For undamped systems it is fully explicit; for damped systems it can be either semi-implicit or fully explicit depending on the choice made for the velocity term and whether the system is linear or nonlinear. For all vibration problems it is conditionally stable. Mathematical details can be found in [1].

The second integrator used is Newmark's integrator. It is a fully implicit second order method. For linear systems (damped or undamped) it is unconditionally stable. For nonlinear systems it appears to be conditionally stable from numerical experiments, however, no analytical proof or range of stability has been determined. The mathematical formulation can be found in [2].

The third, fourth and sixth techniques are all integrators from the class attributed to Gear and Hindmarsh. These techniques belong to a family of fully implicit integrators which have excellent stability properties for many types of stiff problems. For this study, the second order, fourth order and sixth order integrators were chosen (i.e., the second order is method three, the fourth order is method four and the sixth order is method six). All of these integrators are unconditionally stable for the linear vibration systems (damped or undamped) and are conditionally stable for the nonlinear problems. The mathematical formulation of these methods is given in [1].

The fifth method studied is a fully explicit fourth order Runge-Kutta technique attributed to Gill. It is conditionally stable for all systems but possesses extremely good convergence properties. The mathematical details of this method is given in [1].

To investigate the convergence properties, the maximum time step for convergence to 3% accuracy (compared with the analytic solution) was calculated numerically. The results are shown in Figure 1 (for the undamped problem) and Figure 2 (for the damped problem). In both, the central difference method possessed the best convergence properties (i.e., the time step required was greater) of all the second order methods. Newmark's method was second and the second order Gear-Hindmarsh integrator was the worst. Of the two fourth order methods, the Runge-Kutta-Gill exhibited better convergence properties than the fourth order Gear-Hindmarsh integrator. For most problems, the sixth order Gear-Hindmarsh exhibited better convergence properties than the lower order methods, however, in the damped problem, the Runge-Kutta was better for some stiffness ratios.

It is also important to compare relative computer runtimes for the methods as the computer requirements to achieve a given accuracy is ultimately the most important factor for solving a

Figure 1: Time Step Requirements for the Undamped Problem

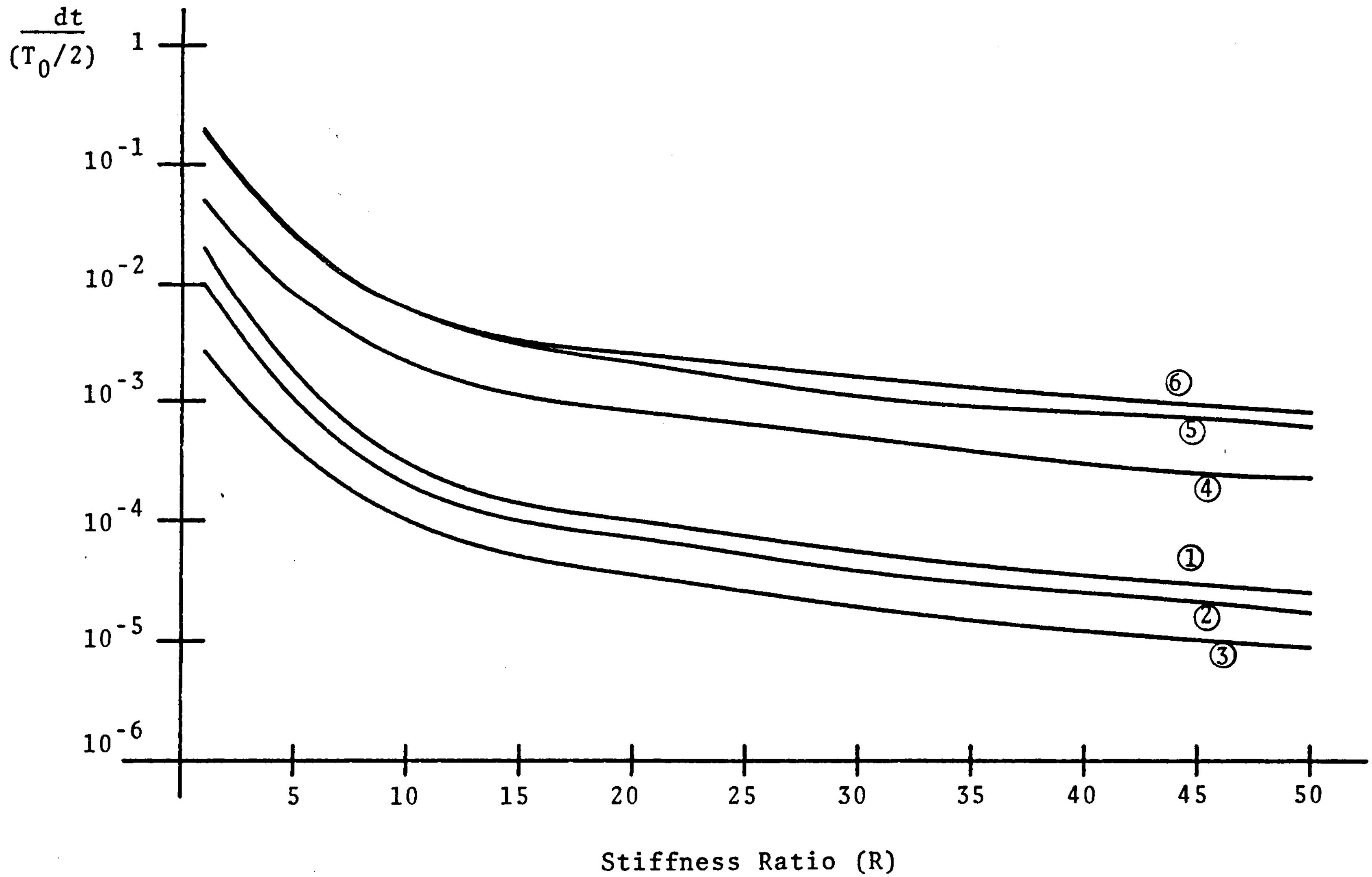
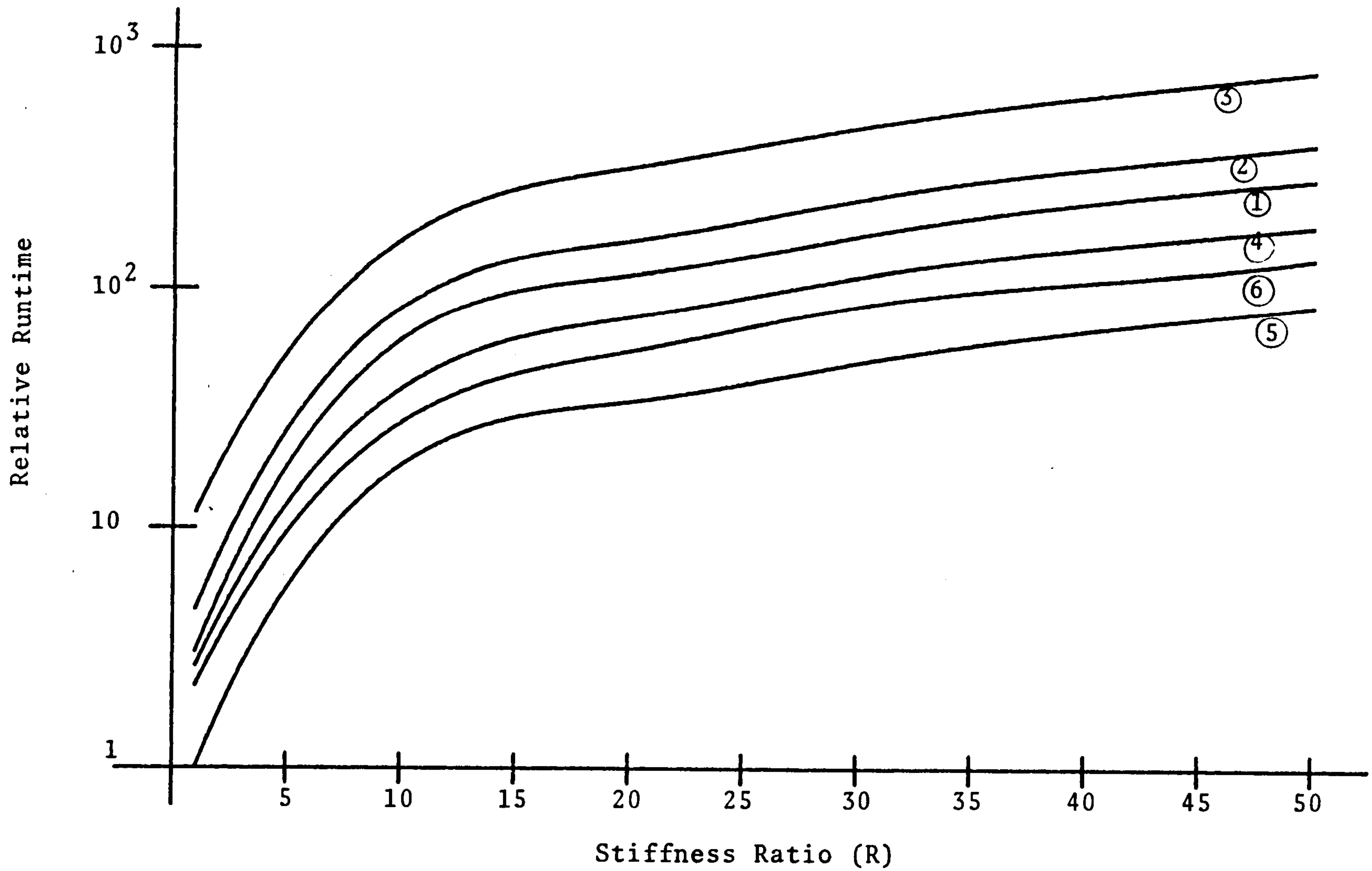


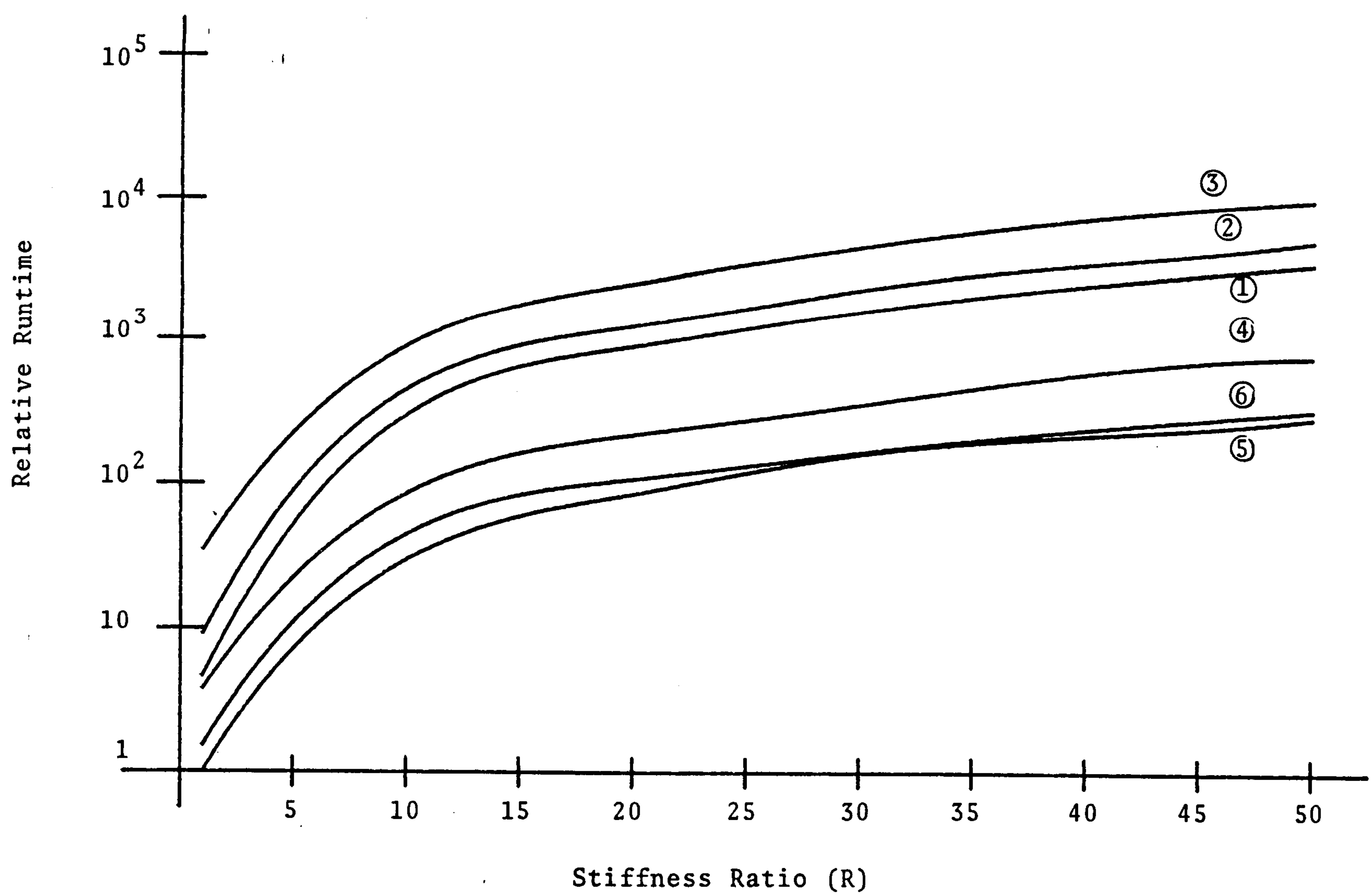
Figure 2: Relative Runtimes for the Damped Problem



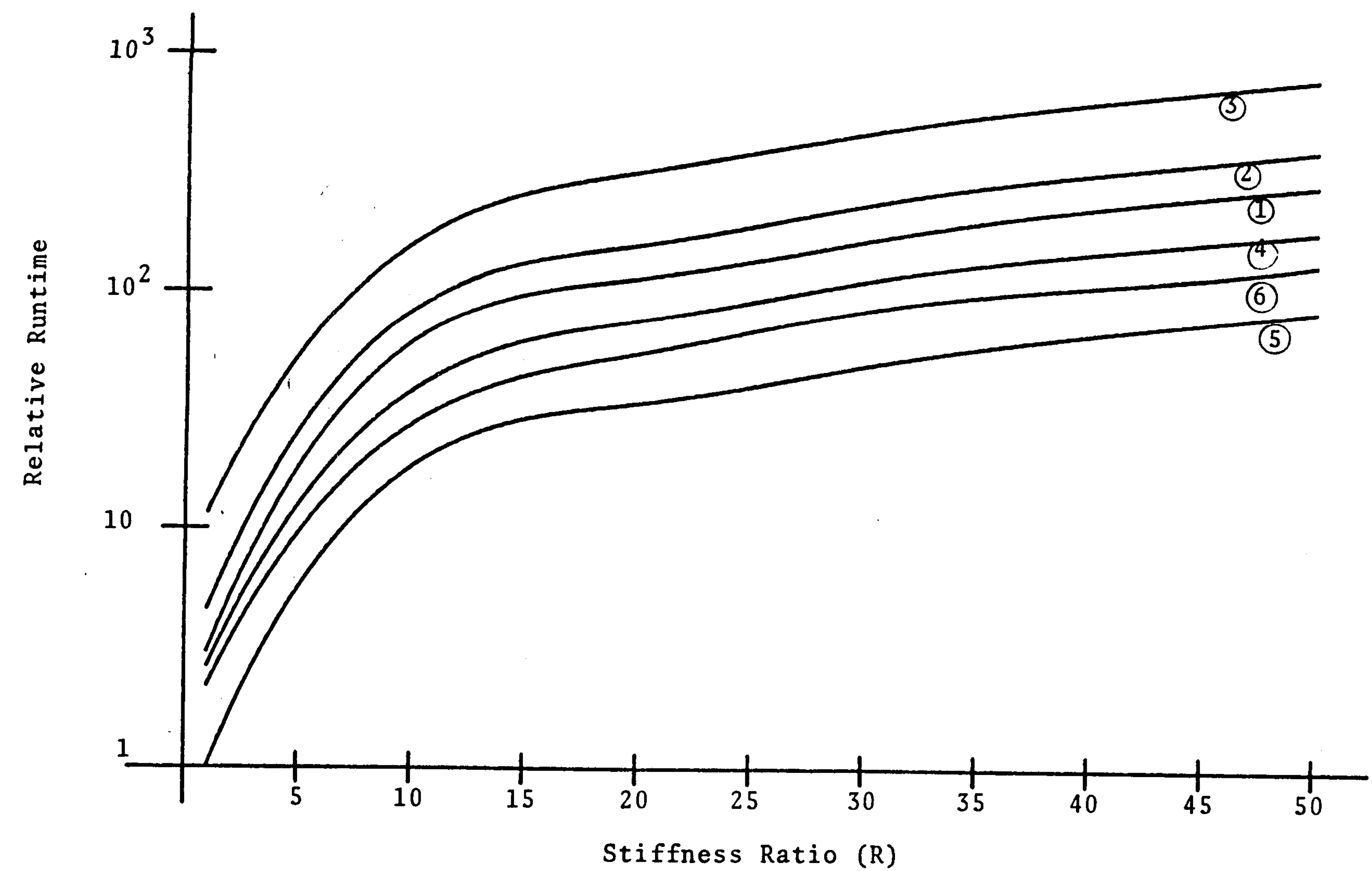
given problem. The relative runtimes are shown in Figure 3 (for the undamped problem) and Figure 4 (for the damped problem). The Runge-Kutta-Gill integrator was the most efficient for all problems studied. It was followed by the sixth order Gear-Hindmarsh, fourth order Gear-Hindmarsh, central difference method, Newmark's method, and second order Gear-Hindmarsh respectively. These results are somewhat contrary to the usual assumptions that implicit integrators outperform explicit for problems which do not exhibit fast transients.

The conclusions of this study are that higher order integrators show great potential for reducing computer solution time for dynamical systems and that implicit integrators may not be the most efficient methods for these systems. The role of stability is grossly overestimated for problems exhibiting significant stiffness (as most real problems do) and large errors may occur from the nonconservative use of implicit integrators.

Relative Runtimes for the Undamped Problem



Relative Runtimes for the Damped Problem



REFERENCES

[1] J. D. Lambert, Computational Methods in Ordinary differential Equations, John Wiley & Sons, London, 1973.

[2] K. J. Bathe, Finite Element Procedures in Engineering Analysis, Prentice Hall, Inc., Englewood Cliffs, NJ, 1982.

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